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Two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric lattice gauge theory with matter fields in the fundamental representation

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Abstract: In this paper, we construct a lattice formulation for two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory with matter fields in the fundamental representation. We first construct it by the orbifolding procedure from Yang-Mills matrix theory with eight supercharges. We show that we can obtain the same lattice formulation by extending the geometrical discretization scheme. This suggests that the equivalence between the two schemes holds even for theories with matter fields.

Keywords: Extended Supersymmetry, Lattice Gauge Field Theories, Lattice Quantum Field Theory, M(atrix) Theories.

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Contents

1. Introduction

Supersymmetric gauge theory is one of the most exciting topics in high energy physics from various points of view. Among many attempts to understand the nature of supersymmetric gauge theory, there have recently been many important developments towards putting exactly preserved supersymmetries on a space-time lattice. In [1]–[3], several lattice formulations have been constructed by the so-called orbifolding procedure from Yang-Mills matrix theories.[1] In [9]–[12], Catterall formulated several lattice theories using a general prescription, the so-called geometrical discretization. In [13]–[16], Sugino discretized topologically twisted gauge theories while keeping the BRST symmetry on a lattice. One characteristic feature of Sugino's lattice formulations is that link variables are expressed by unitary matrices like conventional lattice gauge theories, which is a desirable condition for numerical simulations [17 – 19]. In [20]–[23], the authors constructed lattice theories based on a deformed supersymmetry algebra on a lattice.[2] See [28] for a different approach to lattice supersymmetry in terms of a deformed type IIB matrix model without orbifolding. For other approaches to examine supersymmetric gauge theories on a lattice, see [29]–[38]. For numerical approach to supersymmetric theories without using lattice formulations, see [39, 40].[3]

One of the most important recent results in supersymmetric lattice gauge theory is that the above seemingly different lattice formulations which preserve supersymmetry on a lattice are related to each other. Indeed, the geometrical discretization scheme was found to be equivalent with the orbifolding procedure [45, 12, 46]. We can directly derive the prescription of the geometrical discretization scheme by combining a dimensional reduction and the orbifolding procedure. This means that the geometrical discretization gives an

[1] For further analysis, see, *e.g.*, refs. [4]–[8].

[2] For a discussion on consistency in the deformation of supersymmetry on a lattice, see [24 – 26, 22]. See also [27] for a further discussion on the consistency connecting with large-N limit.

[3] See also [41]–[44] for numerical study of black hole thermodynamics and gauge/gravity duality.

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effective shortcut to the orbifolding procedure. Practically, this equivalence makes it easy to identify the naive continuum limit of a lattice theory since, using this equivalence, we can directly construct the lattice formulation from the continuum theory. In [47], Sugino's lattice formulation was shown to be derived from Catterall's complexified lattice theory [10] by restricting the degrees of freedom of the complexified fields while preserving the supercharge. Furthermore, in [48], the formulations provided by the link approach were also shown to be the same with those of orbifolding.

The purpose of this paper is to construct a lattice theory for two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory with matter fields in the fundamental representation using both the schemes of the orbifolding procedure and the geometrical discretization. In [49], the authors constructed a lattice theory for two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory with matter fields in the *adjoint* representation and suggested that matter fields in the fundamental representation can be introduced by considering an additional Z_2 transformation in the orbifolding procedure. In this paper, we explicitly realize this idea to construct a lattice theory for two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory coupled with matter fields in the fundamental representation from a mother theory with eight supercharges.[4] We derive the same lattice formulation from the continuum theory by slightly extending the geometrical discretization scheme so that we can apply it to a theory with matter fields.

The organization of this paper is as follows. In the next section, we construct a lattice formulation for two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory with hypermultiplets in the fundamental representation using the orbifolding procedure. In section 3, we extend the prescription of the geometrical discretization scheme and derive the same lattice action that is constructed in section 2. Section 4 is devoted to the conclusion and some discussions.

2. Construction of lattice theory via orbifolding procedure

In this section, we construct a lattice theory of two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory coupled with hypermultiplets in the fundamental representation using the orbifolding procedure. We start with a matrix theory (mother theory) with eight supercharges used in [2]:

$$S_{\mathrm{m}} = \mathrm{Tr}\left(\frac{1}{4}|[z_a, z_b]|^2 + \frac{1}{8}[z_a, \bar{z}_a]^2 + \eta[\bar{z}_a, \psi_a] \right.$$
$$\left. + \frac{1}{2}\xi_{ab}\left([z_a, \psi_b] - [z_b, \psi_a]\right) + \frac{1}{2}\chi_{abc}[\bar{z}_a, \xi_{bc}]\right), \tag{2.1}$$

where $a,b,c = 1,2,3$, z_a and $\bar{z}_a$ are bosonic complex matrices and η, ψ_a, ξ_{ab} and χ_{abc} are fermionic complex matrices, which are assumed to be antisymmetric under the permutation of the indices. We also assume that all the matrices are of the size $(N_c+N_f)N^2$. This action is invariant under a global symmetry $\mathrm{SO}(6)\times\mathrm{SU}(2)$ and a gauge symmetry $\mathrm{U}((N_c+$

[4] For an alternative application of this idea, see [50].

	z_1	z_2	z_3	D	η	ξ_{23}	ξ_{31}	ξ_{12}	χ_{123}	ψ_1	ψ_2	ψ_3
q_1	1	0	0	0	1/2	1/2	-1/2	-1/2	1/2	1/2	-1/2	-1/2
q_2	0	1	0	0	1/2	-1/2	1/2	-1/2	1/2	-1/2	1/2	-1/2
q_3	0	0	1	0	1/2	-1/2	-1/2	1/2	1/2	-1/2	-1/2	1/2
q_4	0	0	0	0	1/2	1/2	1/2	1/2	-1/2	-1/2	-1/2	-1/2

Table 1: U(1) charges of the fields in the mother theory.

	z_m	$\bar{z}_m$	ϕ	$\bar{\phi}$	D	η	ψ_m	ξ_{12}	$\bar{\eta}$	$\bar{\psi}_m$	$\bar{\xi}_{12}$
(r_1, r_2)	$\mathbf{e}_m$	$-\mathbf{e}_m$	$\mathbf{q}$	$-\mathbf{q}$	0	0	$\mathbf{e}_m$	$-\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{q}$	$-\mathbf{e}_m - \mathbf{q}$	$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{q}$
s	0	0	1	1	0	0	0	0	1	1	1

Table 2: U(1) charges and "parity" that we use to construct a lattice theory. $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent integer valued two-vectors and $\mathbf{q}$ is a linear combination of $\mathbf{e}_1$ and bfe_2. The U(1) charges of η is tuned to be zero so that the lattice theory preserves one supercharge.

$N_f)N^2$). This is obvious from the fact that (2.1) is obtained by dimensionally reducing the action of six-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory followed by an appropriate field redefinition [2] (see also [8]). For the purpose of later discussion we rewrite $\{z_3, \bar{z}_3, \psi_3, \xi_{m3}, \chi_{123}\}$ as $\{\phi, \bar{\phi}, \bar{\eta}, \bar{\psi}_m, \bar{\xi}_{12}\}$. Then the action (2.1) can be rewritten as

$$\begin{aligned} S_{\mathrm{m}} = \mathrm{Tr}\bigg(& \frac{1}{4}\left|[z_m, z_n]\right|^2 + \frac{1}{8}\left[z_m, \bar{z}_m\right]^2 - \frac{1}{8}D^2 \\ & + \frac{1}{4}\left|[z_m, \phi]\right|^2 + \frac{1}{4}\left|[z_m, \bar{\phi}]\right|^2 - \frac{1}{4}D[\phi, \bar{\phi}] \\ & + \eta[\bar{z}_m, \psi_m] + \frac{1}{2}\xi_{mn}\left([z_m, \psi_n] - [z_n, \psi_m]\right) \\ & + \bar{\eta}[z_m, \bar{\psi}_m] + \frac{1}{2}\bar{\xi}_{mn}\left([\bar{z}_m, \bar{\psi}_n] - [\bar{z}_n, \bar{\psi}_m]\right) \\ & + \eta[\bar{\phi}, \bar{\eta}] - \bar{\psi}_m[\phi, \psi_m] + \frac{1}{2}\bar{\xi}_{mn}[\bar{\phi}, \xi_{mn}]\bigg), \end{aligned} \tag{2.2}$$

where $m, n = 1, 2$ and we have introduced an auxiliary field D. In the orbifolding procedure, the maximal U(1) symmetry, $\mathrm{U}(1)^4$ in this case, plays an important role. In the expression (2.1) or (2.2), the U(1) symmetry is manifest and the charge assignment is given in table 1. In order to construct a two-dimensional lattice theory with at least one preserved supercharge, we define two different charges (r_1, r_2) as two different linear combinations of q_i with requiring $r_1 = r_2 = 0$ for η [8]. In addition to this $\mathrm{U}(1)^2$ symmetry, we further consider a Z_2 symmetry [49] that transforms the fields as $\Phi \to e^{s\pi i}\Phi$ ($s = 0, 1$) corresponding to "parity" associated with the fields. We define s as

$$s \equiv q_3 - q_4 + 1 \pmod 2. \tag{2.3}$$

As a result, the U(1) charges and the parity are summarized as in table 2, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two linearly independent integer valued two-vectors and $\mathbf{q}$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. As shown in [8], the lattice structure is determined not by the detail of the assignment of the U(1) charges but the linear relation between $\mathbf{e}_m$ and $\mathbf{q}$. Therefore we assume $\mathbf{e}_m = \hat{m}$; a unit vector in the positive m'th direction in the following.

Using the U(1) charges and the parity given above, we carry out the following two kinds of orbifold projections. We first define two Z_N^2 transformations generated by

$$\gamma_i : \Phi \to \omega^{r_i} \Omega_i \Phi \Omega_i^{-1}, \quad (i = 1, 2) \tag{2.4}$$

where $\omega = e^{2\pi i/N}$, r_i are U(1) charges of a matrix Φ and Ω_i are defined by

$$\begin{aligned} \Omega_1 &= \Big(\mathbf{1}_{N_c} \otimes U_N \otimes \mathbf{1}_N\Big) \oplus \Big(\mathbf{1}_{N_f} \otimes U_N \otimes \mathbf{1}_N\Big), \\ \Omega_2 &= \Big(\mathbf{1}_{N_c} \otimes \mathbf{1}_N \otimes U_N\Big) \oplus \Big(\mathbf{1}_{N_f} \otimes \mathbf{1}_N \otimes U_N\Big), \end{aligned} \tag{2.5}$$

using the clock matrix, $U_N \equiv \mathrm{diag}(\omega, \omega^2, \ldots, \omega^N)$. In addition, we define a Z_2 transformation generated by

$$p : \Phi \to e^{s\pi i} P \Phi P, \qquad P \equiv \begin{pmatrix} \mathbf{1}_{N_c N^2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{N_f N^2} \end{pmatrix}, \tag{2.6}$$

where $e^{s\pi i}$ is the parity associated with the matrix Φ.

The orbifold projection is defined by projecting out such elements of each matrix that are not invariant under the transformations (2.4) and (2.6). To explain how these projections work, we express a general matrix Φ by four blocks of the size $N_c N^2 \times N_c N^2$, $N_c N^2 \times N_f N^2$, $N_f N^2 \times N_c N^2$ and $N_f N^2 \times N_f N^2$ as

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{2.7}$$

Suppose the U(1) charges of Φ is $\mathbf{r} = (r_1, r_2)$. Then, after the projection associated with (2.4), each block is expressed as

$$\Phi_{ij} = \sum_{\mathbf{k} \in \mathbb{Z}_N^2} \Phi_{ij}(\mathbf{k}) \otimes E_{\mathbf{k}, \mathbf{k}+\mathbf{r}}, \quad (i, j = 1, 2) \tag{2.8}$$

where $E_{\mathbf{k},\mathbf{l}} = E_{k_1,l_1} \otimes E_{k_2,l_2}$ with $(E_{kl})_{mn} = \delta_{km}\delta_{ln}$ and $\Phi_{ij}(\mathbf{k})$ is a matrix with the size $N_c \times N_c$, $N_c \times N_f$, $N_f \times N_c$ and $N_f \times N_f$ corresponding to $(i, j) = (1, 1), (1, 2), (2, 1)$ and $(2, 2)$, respectively. Furthermore, if Φ is parity even(odd), the blocks Φ_{12} and Φ_{21} (Φ_{11} and Φ_{22}) are projected out by (2.6). As a result, after the projections (2.4) and (2.6), Φ with $s = 0$ can be written as

$$\Phi^{(s=0)} = \begin{pmatrix} \sum_{\mathbf{k}} \Phi_{11}(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{r}} & \mathbf{0} \\ \mathbf{0} & \sum_{\mathbf{k}} \Phi_{22}(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{r}} \end{pmatrix}, \tag{2.9}$$

and Φ with $s = 1$ can be written as

$$\Phi^{(s=1)} = \begin{pmatrix} \mathbf{0} & \sum_{\mathbf{k}} \Phi_{12}(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{r}} \\ \sum_{\mathbf{k}} \Phi_{21}(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{r}} & \mathbf{0} \end{pmatrix}. \tag{2.10}$$

We introduce a notation to express (2.9) and (2.10) as

$$\Phi^{(s=0)} = \sum_{\mathbf{k}} \left(\Phi_{11}(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{r}} + \Phi_{22}(\mathbf{k}) \otimes E_{N+\mathbf{k},N+\mathbf{k}+\mathbf{r}}\right), \tag{2.11}$$

and

$$\Phi^{(s=1)} = \sum_{\mathbf{k}} \left(\Phi_{12}(\mathbf{k}) \otimes E_{\mathbf{k},N+\mathbf{k}+\mathbf{r}} + \Phi_{21}(\mathbf{k}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{r}}\right), \tag{2.12}$$

respectively. Then, from the assignment of the U(1) charges and the parity given above, we see that the matrices $\{z_m, \bar{z}_m, D, \eta, \psi_m, \xi_{12}\}$ can be written as

$$\begin{aligned}
z_m &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(z_m(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\hat{m}} + \hat{z}_m(\mathbf{k}) \otimes E_{N+\mathbf{k},N+\mathbf{k}+\hat{m}}\right), \\
\bar{z}_m &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{z}_m(\mathbf{k}) \otimes E_{\mathbf{k}+\hat{m},\mathbf{k}} + \bar{\hat{z}}_m(\mathbf{k}) \otimes E_{N+\mathbf{k}+\hat{m},N+\mathbf{k}}\right), \\
D &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(2d(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}} + 2\hat{d}(\mathbf{k}) \otimes E_{N+\mathbf{k},N+\mathbf{k}}\right), \\
\eta &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\lambda(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}} + \hat{\lambda}(\mathbf{k}) \otimes E_{N+\mathbf{k},N+\mathbf{k}}\right), \\
\psi_m &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\lambda_m(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\hat{m}} + \hat{\lambda}_m(\mathbf{k}) \otimes E_{N+\mathbf{k},N+\mathbf{k}+\hat{m}}\right), \\
\xi_{mn} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\lambda_{mn}(\mathbf{k}) \otimes E_{\mathbf{k}+\hat{m}+\hat{n},\mathbf{k}} + \hat{\lambda}_{mn}(\mathbf{k}) \otimes E_{N+\mathbf{k}+\hat{m}+\hat{n},N+\mathbf{k}}\right),
\end{aligned} \tag{2.13}$$

and the others can be written as

$$\begin{aligned}
\phi &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(-\sqrt{2}i\bar{\tilde{\phi}}(\mathbf{k}) \otimes E_{\mathbf{k},N+\mathbf{k}+\mathbf{q}} + \sqrt{2}i\bar{\phi}(\mathbf{k}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{q}}\right), \\
\bar{\phi} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(-\sqrt{2}i\phi(\mathbf{k}+\mathbf{q}) \otimes E_{\mathbf{k}+\mathbf{q},N+\mathbf{k}} + \sqrt{2}i\tilde{\phi}(\mathbf{k}) \otimes E_{N+\mathbf{k}+\mathbf{q},\mathbf{k}}\right), \\
\bar{\eta} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{\tilde{\psi}}(\mathbf{k}) \otimes E_{\mathbf{k},N+\mathbf{k}+\mathbf{q}} + \bar{\psi}(\mathbf{k}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{q}}\right), \\
\bar{\psi}_m &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\psi_m(\mathbf{k}+\hat{m}+\mathbf{q}) \otimes E_{\mathbf{k}+\hat{m}+\mathbf{q},N+\mathbf{k}} + \tilde{\psi}_m(\mathbf{k}) \otimes E_{N+\mathbf{k}+\hat{m}+\mathbf{q},\mathbf{k}}\right), \\
\bar{\xi}_{mn} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{\tilde{\psi}}_{mn}(\mathbf{k}) \otimes E_{\mathbf{k},N+\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}} + \bar{\psi}_{mn}(\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}}\right).
\end{aligned} \tag{2.14}$$

The action of the orbifold lattice theory is obtained by substituting (2.13) and (2.14) into the action of the mother theory (2.2) and expand it around a classical configuration,

$$z_m(\mathbf{k}) = \bar{z}_m(\mathbf{k}) = \frac{1}{a}\mathbf{1}_{N_c}, \quad \hat{z}_m(\mathbf{k}) = \bar{\hat{z}}_m(\mathbf{k}) = \frac{1}{a}\mathbf{1}_{N_f}. \tag{2.15}$$

Note that the obtained theory preserves only one supercharge for any value of $\mathbf{q}$. It is obvious from the fact that only one of the eight supersymmetry parameters in the mother

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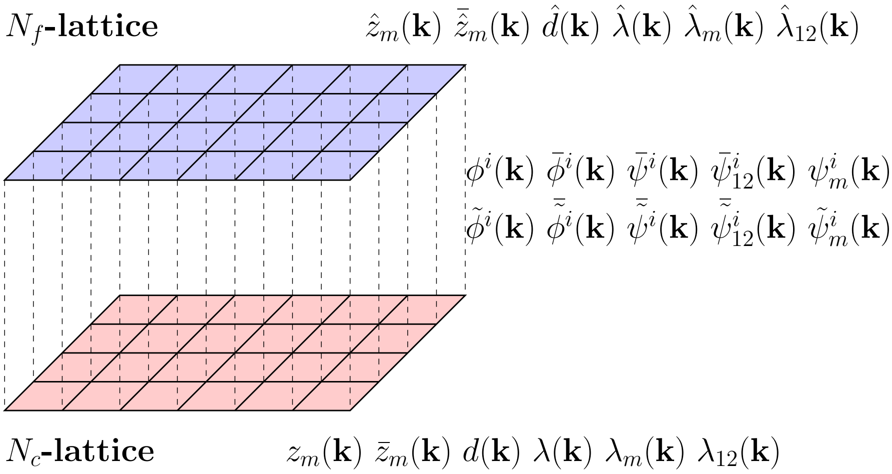

Figure 1: The lattice space-time obtained by the two kinds of orbifold projections corresponding to (2.4) and (2.6). The lattice variables $\{z_m(\mathbf{k}),\ \bar{z}_m(\mathbf{k}),\ d(\mathbf{k}),\ \lambda(\mathbf{k}),\ \lambda_m(\mathbf{k}),\ \lambda_{12}(\mathbf{k})\}$ live on the N_c-lattice and $\{\hat{z}_m(\mathbf{k}),\ \bar{\hat{z}}_m(\mathbf{k}),\ \hat{d}(\mathbf{k}),\ \hat{\lambda}(\mathbf{k}),\ \hat{\lambda}_m(\mathbf{k}),\ \hat{\lambda}_{12}(\mathbf{k})\}$ live on the N_f-lattice. The matter fields $\{\phi^i(\mathbf{k}),\ \bar{\bar{\phi}}^i(\mathbf{k}),\ \bar{\bar{\psi}}^i(\mathbf{k}),\ \bar{\bar{\psi}}^i_{12}(\mathbf{k}),\ \psi^i_m(\mathbf{k})\}$ and $\{\tilde{\bar{\phi}}^i(\mathbf{k}),\ \tilde{\bar{\psi}}^i(\mathbf{k}),\ \tilde{\bar{\psi}}^i(\mathbf{k}),\ \tilde{\bar{\psi}}^i{}_{12}(\mathbf{k}),\ \tilde{\bar{\psi}}^i_m(\mathbf{k})\}$ live on links connecting the two lattice space-times.

theory has the charge assignment $\mathbf{r} = 0$ and $s = 0$. Since the supersymmetry parameters are c-numbers, the other seven supersymmetry parameters are projected out by the orbifold projections [48]. The preserved supersymmetry transformation on the lattice is obtained by substituting (2.13) and (2.14) into the supersymmetry transformation of the matrices in the mother theory,

$$\begin{aligned} Qz_m &= \psi_m, & Q\bar{z}_m &= 0, \quad QD = [\psi_m, \bar{z}_m], \\ Q\eta &= \frac{1}{4}\left([z_m,\bar{z}_m] - D\right), & Q\psi_m &= 0, \quad Q\xi_{mn} = \frac{1}{2}[\bar{z}_m, \bar{z}_n], \\ Q\phi &= \bar{\eta}, & Q\bar{\phi} &= 0, \quad Q\bar{\eta} = 0, \quad Q\bar{\psi}_m = \frac{1}{2}[\bar{z}_m, \bar{\phi}], \quad Q\bar{\xi}_{mn} = 0, \end{aligned} \tag{2.16}$$

followed by the shift, $z_m \to 1/a + z_m$ and $\bar{z}_m \to 1/a + \bar{z}_m$. We can explicitly see the preserved supersymmetry by rewriting the action of the mother theory (2.2) in a Q-exact form:

$$S_m = Q\mathrm{Tr}\left\{\frac{1}{2}\eta\left([z_m,\bar{z}_m] + 2[\phi,\bar{\phi}] + D\right) - \frac{1}{2}\xi_{mn}[z_m,z_n] - \bar{\psi}_m[z_m,\phi] - \frac{1}{2}\bar{\xi}_{mn}F_{mn}\right\}, \tag{2.17}$$

where we have introduced an auxiliary field F_{mn} whose supersymmetry transformation is given by

$$QF_{mn} = [\bar{z}_m, \bar{\psi}_n] - [\bar{z}_n, \bar{\psi}_m] - [\xi_{mn}, \bar{\phi}]. \tag{2.18}$$

From the construction, the obtained theory is a supersymmetric lattice gauge theory with a gauge group $\mathrm{U}(N_c) \times \mathrm{U}(N_f)$ defined on two copies of two-dimensional lattice space-times of size N^2 (figure 1). The lattice variables $\{z_m(\mathbf{k}),\ \bar{z}_m(\mathbf{k}),\ d(\mathbf{k}),\ \lambda(\mathbf{k}),\ \lambda_m(\mathbf{k}),\ \lambda_{12}(\mathbf{k})\}$ and $\{\hat{z}_m(\mathbf{k}),\ \bar{\hat{z}}_m(\mathbf{k}),\ \hat{d}(\mathbf{k}),\ \hat{\lambda}(\mathbf{k}),\ \hat{\lambda}_m(\mathbf{k}),\ \hat{\lambda}_{12}(\mathbf{k})\}$ transform in the representation (adj, $\mathbf{1}$) and ($\mathbf{1}$, adj) of $\mathrm{U}(N_c) \times \mathrm{U}(N_f)$, respectively, and live on different lattice

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space-times. In the following, we call these two lattice space-times the N_c-lattice and the N_f-lattice, respectively. On the other hand, $\{\phi^i(\mathbf{k}),\ \bar{\tilde{\phi}}^i(\mathbf{k}),\ \bar{\tilde{\psi}}^i(\mathbf{k}),\ \bar{\tilde{\psi}}^i_{12}(\mathbf{k}),\ \psi^i_m(\mathbf{k})\}$ and $\{\bar{\phi}^i(\mathbf{k}),\ \tilde{\psi}^i(\mathbf{k}),\ \tilde{\psi}^i(\mathbf{k}),\ \tilde{\psi}^i{}_{12}(\mathbf{k}),\ \bar{\psi}^i_m(\mathbf{k})\}$ transform in the representation $(\Box,\overline{\Box})$ and $(\overline{\Box},\Box)$ of $\mathrm{U}(N_c)\times\mathrm{U}(N_f)$, respectively, and live on links connecting the two lattice space-times. Although this lattice action for this quiver gauge theory is not our main purpose, it is an important result. We write down the action in appendix A.

Finally, in order to construct a lattice theory with matter fields in the fundamental representation of the gauge group $\mathrm{U}(N_c)$, we make the fields living on the N_f-lattice non-dynamical by hand:

$$\hat{z}_m(\mathbf{k}) = \hat{\bar{z}}_m(\mathbf{k}) = \hat{d}(\mathbf{k}) = \hat{\lambda}(\mathbf{k}) = \hat{\lambda}_m(\mathbf{k}) = \hat{\lambda}_{12}(\mathbf{k}) = 0. \tag{2.19}$$

The restricted theory is still supersymmetric since this restriction does not conflict with the supersymmetry (2.16). By this operation, the symmetry $\mathrm{U}(N_f)$ is no longer a gauge symmetry but a flavor symmetry and we obtain a lattice action with $\mathrm{U}(N_c)$ gauge symmetry. Finally we obtain the lattice action:

$$S_{\text{lat}} = S_{\text{gauge}} + S_{\text{matter}}, \tag{2.20}$$

with

$$\begin{aligned}
S_{\text{gauge}} =& \mathrm{Tr}_{N_c}\sum_{\mathbf{k}}\Bigg(\frac{1}{4}\left|\nabla^+_m z_n(\mathbf{k}) - \nabla^+_n z_m(\mathbf{k}) + z_m(\mathbf{k})z_n(\mathbf{k}+\hat{m}) - z_n(\mathbf{k})z_m(\mathbf{k}+\hat{n})\right|^2 \\
&+\frac{1}{8}\left(\nabla^-_m\left(z_m(\mathbf{k})+\bar{z}_m(\mathbf{k})\right) + z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\hat{m})z_m(\mathbf{k}-\hat{m})\right)^2 \\
&-\frac{1}{2}d(\mathbf{k})^2 - \lambda(\mathbf{k})\bar{\mathcal{D}}^-_m\lambda_m(\mathbf{k}) + \frac{1}{2}\lambda_{mn}(\mathbf{k})\left(\mathcal{D}^+_m\lambda_n(\mathbf{k}) - \mathcal{D}^+_n\lambda_m(\mathbf{k})\right)\Bigg),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
S_{\text{matter}} =& \sum_{\mathbf{k}}\Bigg(\frac{1}{2}\bar{\mathcal{D}}^+_m\bar{\phi}^i(\mathbf{k})\mathcal{D}^+_m\phi^i(\mathbf{k}) + \frac{1}{2}\mathcal{D}^+_m\bar{\phi}^i(\mathbf{k})\bar{\mathcal{D}}^+_m\phi^i(\mathbf{k}) \\
&+\frac{1}{2}\bar{\mathcal{D}}^+_m\tilde{\phi}^i(\mathbf{k})\mathcal{D}^+_m\bar{\tilde{\phi}}^i(\mathbf{k}) + \frac{1}{2}\mathcal{D}^+_m\tilde{\phi}^i(\mathbf{k})\bar{\mathcal{D}}^+_m\bar{\tilde{\phi}}^i(\mathbf{k}) \\
&+\bar{\phi}^i(\mathbf{k})d(\mathbf{k})\phi^i(\mathbf{k}) - \tilde{\phi}^i(\mathbf{k})d(\mathbf{k})\bar{\tilde{\phi}}^i(\mathbf{k}) \\
&+\bar{\psi}^i(\mathbf{k})\mathcal{D}^-_m\psi^i_m(\mathbf{k}+\hat{m}) + \tilde{\psi}^i_m(\mathbf{k})\mathcal{D}^-_m\bar{\tilde{\psi}}^i(\mathbf{k}+\hat{m}) \\
&-\frac{1}{2}\bar{\psi}^i_{mn}(\mathbf{k}+\hat{m}+\hat{n})\left[\bar{\mathcal{D}}^+_m\psi^i_n(\mathbf{k}+\hat{n}) - \bar{\mathcal{D}}^+_n\psi^i_m(\mathbf{k}+\hat{m})\right] \\
&-\frac{1}{2}\left[\tilde{\psi}^i_n(\mathbf{k}+\hat{m})\bar{\mathcal{D}}^+_m\bar{\tilde{\psi}}^i_{mn}(\mathbf{k}) - \tilde{\psi}^i_m(\mathbf{k}+\hat{n})\bar{\mathcal{D}}^+_n\bar{\tilde{\psi}}^i_{mn}(\mathbf{k}))\right] \\
&+\sqrt{2}i\left(\bar{\psi}^i(\mathbf{k})\lambda(\mathbf{k})\phi^i(\mathbf{k}) - \tilde{\phi}^i(\mathbf{k})\lambda(\mathbf{k})\bar{\tilde{\psi}}^i(\mathbf{k})\right) \\
&+\sqrt{2}i\left(-\tilde{\psi}^i_m(\mathbf{k})\lambda_m(\mathbf{k})\bar{\tilde{\phi}}^i(\mathbf{k}+\hat{m}) + \bar{\phi}^i(\mathbf{k})\lambda_m(\mathbf{k})\psi^i_m(\mathbf{k}+\hat{m})\right) \\
&+\frac{\sqrt{2}i}{2}\left(\bar{\psi}^i_{mn}(\mathbf{k}+\hat{m}+\hat{n})\lambda_{mn}(\mathbf{k})\phi^i(\mathbf{k}) - \tilde{\phi}^i(\mathbf{k}+\hat{m}+\hat{n})\lambda_{mn}(\mathbf{k})\bar{\tilde{\psi}}^i_{mn}(\mathbf{k})\right)\Bigg),
\end{aligned} \tag{2.22}$$

where $i = 1, \ldots, N_f$ is a flavor index, $\mathcal{D}^{\pm}_m$ and $\bar{\mathcal{D}}^{\pm}_m$ are covariant difference defined as

$$
\begin{aligned}
&\mathcal{D}^{+}_m \Phi_{\mathrm{adj}}(\mathbf{k}) \equiv \nabla^{+}_m \Phi_{\mathrm{adj}}(\mathbf{k}) + z_m(\mathbf{k})\Phi_{\mathrm{adj}}(\mathbf{k}+\hat{m}) - \Phi_{\mathrm{adj}}(\mathbf{k}) z_m(\mathbf{k}+\mathbf{r}),\\
&\bar{\mathcal{D}}^{+}_m \Phi_{\mathrm{adj}}(\mathbf{k}) \equiv \nabla^{+}_m \Phi_{\mathrm{adj}}(\mathbf{k}) + \Phi_{\mathrm{adj}}(\mathbf{k}+\hat{m})\bar{z}_m(\mathbf{k}+\mathbf{r}) - \bar{z}_m(\mathbf{k})\Phi_{\mathrm{adj}}(\mathbf{k}),\\
&\mathcal{D}^{-}_m \Phi_{\mathrm{adj}}(\mathbf{k}) \equiv \mathcal{D}^{+}_m \Phi_{\mathrm{adj}}(\mathbf{k}-\hat{m}), \quad \bar{\mathcal{D}}^{-}_m \Phi_{\mathrm{adj}}(\mathbf{k}) \equiv \bar{\mathcal{D}}^{+}_m \Phi_{\mathrm{adj}}(\mathbf{k}-\hat{m}),
\end{aligned} \tag{2.23}
$$

for a lattice field in the adjoint representation living on a link $(\mathbf{k}, \mathbf{k}+\mathbf{r})$,

$$
\begin{aligned}
&\mathcal{D}^{+}_m \Phi_{\Box}(\mathbf{k}) \equiv \nabla^{+}_m \Phi_{\Box}(\mathbf{k}) + z_m(\mathbf{k})\Phi_{\Box}(\mathbf{k}+\hat{m}),\\
&\bar{\mathcal{D}}^{+}_m \Phi_{\Box}(\mathbf{k}) \equiv \nabla^{+}_m \Phi_{\Box}(\mathbf{k}) - \bar{z}_m(\mathbf{k})\Phi_{\Box}(\mathbf{k}),\\
&\mathcal{D}^{-}_m \Phi_{\Box}(\mathbf{k}) \equiv \mathcal{D}^{+}_m \Phi_{\Box}(\mathbf{k}-\hat{m}), \quad \bar{\mathcal{D}}^{-}_m \Phi_{\Box}(\mathbf{k}) \equiv \bar{\mathcal{D}}^{+}_m \Phi_{\Box}(\mathbf{k}-\hat{m}),
\end{aligned} \tag{2.24}
$$

for a lattice field in the fundamental representation, and

$$
\begin{aligned}
&\mathcal{D}^{+}_m \Phi_{\bar{\Box}}(\mathbf{k}) \equiv \nabla^{+}_m \Phi_{\bar{\Box}}(\mathbf{k}) - \Phi_{\bar{\Box}}(\mathbf{k}) z_m(\mathbf{k}),\\
&\bar{\mathcal{D}}^{+}_m \Phi_{\bar{\Box}}(\mathbf{k}) \equiv \nabla^{+}_m \Phi_{\bar{\Box}}(\mathbf{k}) + \Phi_{\bar{\Box}}(\mathbf{k}+\hat{m})\hat{z}_m(\mathbf{k}),\\
&\mathcal{D}^{-}_m \Phi_{\bar{\Box}}(\mathbf{k}) \equiv \mathcal{D}^{+}_m \Phi_{\bar{\Box}}(\mathbf{k}-\hat{m}), \quad \bar{\mathcal{D}}^{-}_m \Phi_{\bar{\Box}}(\mathbf{k}) \equiv \bar{\mathcal{D}}^{+}_m \Phi_{\bar{\Box}}(\mathbf{k}-\hat{m}),
\end{aligned} \tag{2.25}
$$

for a lattice field in the anti-fundamental representation. Here, $\nabla^{\pm}_m$ are forward and backward differences defined by

$$
\nabla^{+}_m f(\mathbf{k}) = \frac{1}{a}\left(f(\mathbf{k}+\hat{m}) - f(\mathbf{k})\right), \quad \nabla^{-}_m f(\mathbf{k}) = \frac{1}{a}\left(f(\mathbf{k}) - f(\mathbf{k}-\hat{m})\right). \tag{2.26}
$$

The gauge part of the action (2.21) is nothing but the lattice action for two-dimensional $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills theory given in [1]. On the other hand, as we see in the next section, the continuum limit of (2.22) is the action of hypermultiplets of two-dimensional $\mathcal{N} = (2,2)$ supersymmetry. Using the language of four-dimensional $\mathcal{N} = 1$ supersymmetry, it is obtained by a dimensional reduction to two dimensions from the action,

$$
S^{\mathrm{matter}}_{\mathrm{4D}} = \int d^4x d^2\theta d^2\bar{\theta}\Big(\bar{\Phi}^i e^{2V}\Phi^i + \tilde{\Phi}^i e^{-2V}\bar{\tilde{\Phi}}\Big), \tag{2.27}
$$

where Φ^i and $\tilde{\Phi}^i$ are four-dimensional $\mathcal{N} = 1$ chiral superfields in the fundamental and anti-fundamental representations, respectively. Therefore, the action (2.20) gives a supersymmetric lattice formulation for two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory coupled with hypermultiplets in the fundamental representation. The correspondence between lattice variables and continuum fields becomes clear in the next section. Note that the matter fermions in this lattice theory do not have doublers. In fact, the fermion determinant is proportional to $\nabla^{+}_m \nabla^{-}_m$, which has zero only at the origin of the momentum space.

We close this section by making two comments. First, the matter fields in this lattice theory live only on sites even if they have non-zero U(1) charges in general. Although it seems peculiar at first sight, we can understand it by seeing that we have two lattice space-times and matter fields live on links between the N_c-lattice and the N_f-lattice. Since the N_f-lattice becomes invisible by the operation (2.19), the matter fields behave as site variables on the N_c-lattice.

Second, we can consistently truncate the matter fields with tilde from the action (2.22). This corresponds to the configuration,

$$\begin{aligned}
\phi &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\sqrt{2}i\bar{\phi}(\mathbf{k}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{q}}\right), \quad \bar{\phi} \equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(-\sqrt{2}i\phi(\mathbf{k}+\mathbf{q}) \otimes E_{\mathbf{k}+\mathbf{q},N+\mathbf{k}}\right), \\
\bar{\eta} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{\psi}(\mathbf{k}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{q}}\right), \qquad \bar{\psi}_m \equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\psi_m(\mathbf{k}+\hat{m}+\mathbf{q}) \otimes E_{\mathbf{k}+\hat{m}+\mathbf{q},N+\mathbf{k}}\right), \\
\bar{\xi}_{mn} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{\psi}_{mn}(\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}}\right), \qquad (2.28)
\end{aligned}$$

instead of (2.14). In the language of four-dimensional $N=1$ theory, this is a dimensionally reduced theory of

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\bar{\Phi}^i e^{2V}\Phi^i. \qquad (2.29)$$

3. Derivation via geometrical discretization

In this section, we derive the same lattice action (2.20) by extending the geometrical discretization scheme constructed by Catterall.

In the geometrical discretization scheme, we construct a lattice theory from a continuum theory. Since it is known that the gauge part of the lattice action (2.21) can be obtained by the geometrical discretization scheme [12], we concentrate on the matter action (2.22). In order to write down the continuum action, we start with the action of four-dimensional Euclidean $\mathcal{N}=1$ theory (2.27), which is written down in component as

$$\begin{aligned}
\mathcal{L}_{\rm 4D}^{\rm matter} =& D_\mu\bar{\phi}^i(x)D_\mu\phi^i(x) + D_\mu\tilde{\phi}^i(x)D_\mu\bar{\tilde{\phi}}^i(x) + \bar{\phi}^i(x)d(x)\phi^i(x) - \tilde{\phi}^i(x)d(x)\bar{\tilde{\phi}}^i(x) \\
& - i\bar{\xi}^i(x)\bar{\sigma}^\mu D_\mu\xi^i(x) - i\tilde{\xi}^i(x)\sigma^\mu D_\mu\bar{\tilde{\xi}}^i(x) \\
& + i\sqrt{2}\left(\bar{\phi}^i(x)\lambda(x)\xi^i(x) - \bar{\xi}^i(x)\bar{\lambda}(x)\phi^i(x)\right) \\
& + i\sqrt{2}\left(\tilde{\phi}^i(x)\bar{\lambda}(x)\bar{\tilde{\xi}}^i(x) - \tilde{\xi}^i(x)\lambda\bar{\tilde{\phi}}^i(x)\right), \qquad (3.1)
\end{aligned}$$

where $\mu = 1,\ldots,4$, D_μ is a four-dimensional covariant derivative, $\{\phi^i(x), \bar{\tilde{\phi}}^i(x)\}$ and $\{\bar{\phi}^i(x), \tilde{\phi}^i(x)\}$ are complex scalar fields in the fundamental and anti-fundamental representations, respectively, $\lambda(x)$ and $\bar{\lambda}(x)$ are two-component spinors in adjoint representation, $\xi^i(x)$ and $\bar{\tilde{\xi}}^i(x)$ are two-component spinors in the fundamental representation, and $\bar{\xi}^i(x)$ and $\tilde{\xi}^i(x)$ are two-component spinors in the anti-fundamental representation.[5]

We next dimensionally reduce (3.1) to two dimensions spanned by $\{x^4, x^2\}$. Correspondingly, we rename $\{v_4, v_2\}$ and $\{v_3, v_1\}$ as $\{A_1, A_2\}$ and $\{\varphi_1, \varphi_2\}$, respectively, and

[5] The notation is based on [51].

write the component of the spinors as

$$
\lambda_\alpha(x) = \begin{pmatrix} \lambda_1(x) \\ \lambda_2(x) \end{pmatrix}, \qquad \bar{\lambda}^{\dot\alpha}(x) = \begin{pmatrix} \lambda_{12}(x) \\ -\lambda(x) \end{pmatrix}, \qquad \xi^i_\alpha(x) = \begin{pmatrix} -\psi^i_2(x) \\ \psi^i_1(x) \end{pmatrix},
$$
$$
\bar{\xi}^{i\dot\alpha}(x) = \begin{pmatrix} \bar{\psi}^i(x) \\ -\bar{\psi}^i_{12}(x) \end{pmatrix}, \qquad \tilde{\xi}^i_\alpha(x) = \begin{pmatrix} -\tilde{\psi}^i_2(x) \\ \tilde{\psi}^i_1(x) \end{pmatrix}, \qquad \bar{\tilde{\xi}}^{i\dot\alpha}(x) = \begin{pmatrix} \bar{\tilde{\psi}}^i(x) \\ -\bar{\tilde{\psi}}^i_{12}(x) \end{pmatrix}. \tag{3.2}
$$

Then, after integrating out the auxiliary field, we obtain the following expression of two-dimensional matter action:

$$
\begin{aligned}
S_{\text{matter}} = \int d^2x \biggl(& \frac{1}{2}\mathcal{D}_m \bar{\phi}^i(x) \bar{\mathcal{D}}_m \phi^i(x) + \frac{1}{2}\bar{\mathcal{D}}_m \bar{\phi}^i(x) \mathcal{D}_m \phi^i(x) \\
& + \frac{1}{2}\mathcal{D}_m \tilde{\phi}^i(x) \bar{\mathcal{D}}_m \bar{\tilde{\phi}}^i(x) + \frac{1}{2}\bar{\mathcal{D}}_m \tilde{\phi}^i(x) \mathcal{D}_m \bar{\tilde{\phi}}^i(x) \\
& + \bar{\phi}^i(x) d(x) \phi^i(x) - \tilde{\phi}^i(x) d(x) \bar{\tilde{\phi}}^i(x) \\
& + \bar{\psi}^i(x) \mathcal{D}_m \psi^i_m(x) + \tilde{\psi}^i(x) \mathcal{D}_m \bar{\tilde{\psi}}^i_m(x) \\
& - \frac{1}{2}\bigl[\bar{\psi}^i_{mn}(x) \bar{\mathcal{D}}_m \psi^i_n(x) - \bar{\psi}^i_{mn}(x) \bar{\mathcal{D}}_n \psi^i_m(x)\bigr] \\
& - \frac{1}{2}\bigl[\tilde{\psi}^i_n(x) \bar{\mathcal{D}}_m \bar{\tilde{\psi}}^i_{mn}(x) - \tilde{\psi}^i_m(x) \bar{\mathcal{D}}_n \bar{\tilde{\psi}}^i_{mn}(x)\bigr] \\
& + \sqrt{2}i\bigl(\bar{\psi}^i(x) \lambda(x) \phi^i(x) - \tilde{\phi}^i(x) \lambda(x) \bar{\tilde{\psi}}^i(x)\bigr) \\
& + \sqrt{2}i\bigl(-\tilde{\psi}^i_m(x) \lambda_m(x) \bar{\tilde{\phi}}^i(x) + \bar{\phi}^i(x) \lambda_m(x) \psi^i_m(x)\bigr) \\
& + \frac{\sqrt{2}i}{2}\bigl(\bar{\psi}^i_{mn}(x) \lambda_{mn}(x) \phi^i(x) - \tilde{\phi}^i(x) \lambda_{mn}(x) \bar{\tilde{\psi}}^i_{mn}(x)\bigr) \biggr),
\end{aligned} \tag{3.3}
$$

where we have defined $\mathcal{D}_m = \partial_m + iA_m + \phi_m$ and $\bar{\mathcal{D}}_m = \partial_m + iA_m - \phi_m$ [12]. For a later discussion, we further rewrite (3.3) in a Q-exact form as

$$
\begin{aligned}
S_{\text{matter}} = \int d^2x Q \biggl(& -i\sqrt{2}\bar{\phi}^i(x) \mathcal{D}_m \psi_m(x) + 2\bar{\phi}^i(x) \lambda(x) \phi^i(x) - \frac{1}{2}\bar{\psi}^i_{mn}(x) f^i_{mn}(x) \\
& - i\sqrt{2}\tilde{\psi}_m(x) \mathcal{D}_m \bar{\tilde{\phi}}^i(x) - 2\tilde{\phi}^i(x) \lambda(x) \bar{\tilde{\phi}}^i(x) - \frac{1}{2}\tilde{f}^i_{mn}(x) \bar{\tilde{\psi}}^i(x) \biggr),
\end{aligned} \tag{3.4}
$$

where the transformation by the supercharge Q is defined by

$$
\begin{aligned}
& Q\mathcal{D}_m(x) = \lambda_m(x), \quad Q\bar{\mathcal{D}}_m(x) = 0, && Qd(x) = \frac{1}{2}\bar{\mathcal{D}}_m \lambda_m(x), \\
& Q\lambda(x) = -\frac{1}{4}[\mathcal{D}_m, \bar{\mathcal{D}}_m](x) - \frac{1}{2}d(x), && Q\lambda_m(x) = 0, \quad Q\lambda_{mn}(x) = \frac{1}{2}[\bar{\mathcal{D}}_m, \bar{\mathcal{D}}_n], \\
& Q\phi^i(x) = 0, \qquad Q\bar{\phi}^i(x) = \frac{1}{i\sqrt{2}}\bar{\psi}^i(x), && Q\tilde{\phi}^i(x) = 0, \qquad Q\bar{\tilde{\phi}}^i(x) = \frac{-1}{i\sqrt{2}}\bar{\tilde{\psi}}^i(x), \\
& Q\bar{\psi}^i(x) = 0, \qquad Q\bar{\psi}^i_{mn}(x) = 0, && Q\psi^i_m(x) = \frac{-1}{i\sqrt{2}}\bar{\mathcal{D}}_m \phi^i(x), \\
& Q\bar{\tilde{\psi}}^i(x) = 0, \qquad Q\bar{\tilde{\psi}}^i_{mn}(x) = 0, && Q\tilde{\psi}_m(x) = \frac{1}{i\sqrt{2}}\bar{\mathcal{D}}_m \tilde{\phi}^i(x),
\end{aligned}
$$

$$
\begin{aligned}
Qf^i_{mn}(x) &= -\bar{\mathcal{D}}_m\psi^i_n(x) + \bar{\mathcal{D}}_n\psi^i_m(x) + i\sqrt{2}\lambda_{mn}(x)\phi^i(x), \\
Q\tilde{f}^i_{mn}(x) &= -\bar{\mathcal{D}}_m\tilde{\psi}^i_n(x) + \bar{\mathcal{D}}_n\tilde{\psi}^i_m(x) + i\sqrt{2}\tilde{\phi}^i(x)\lambda_{mn}(x).
\end{aligned} \tag{3.5}
$$

The prescription of the geometrical discretization scheme given in [10] is summarized as the following four rules:

1. An adjoint p-form field is mapped to a lattice variable on a p-cell. The gauge transformation for a p-form field $f_{\mu_1\cdots\mu_p}(\mathbf{k})$ is given by $f_{\mu_1\cdots\mu_p}(\mathbf{k}) \to g(\mathbf{k})f_{\mu_1\cdots\mu_p}(\mathbf{k})g^{-1}(\mathbf{k}+\hat{\mu}_1+\cdots+\hat{\mu}_p)$ or $f_{\mu_1\cdots\mu_p}(\mathbf{k}) \to g(\mathbf{k}+\hat{\mu}_1+\cdots+\hat{\mu}_p)f_{\mu_1\cdots\mu_p}(\mathbf{k})g^{-1}(\mathbf{k})$ depending on the direction of the p-cell.[6]

2. A curl-like covariant differential is mapped to a covariant forward difference.

3. A divergent-like covariant differential is mapped to a covariant backward difference.

4. An interaction term is written so that it forms a loop on a lattice.

Since this scheme is originally constructed for a theory that contains only adjoint fields, we add the following rule for fields in the fundamental representation:

5. A field in the fundamental or anti-fundamental representation is mapped on a site. The gauge transformation is $\psi(\mathbf{k}) \to g(\mathbf{k})\psi(\mathbf{k})$ and $\bar{\psi}(\mathbf{k}) \to \bar{\psi}(\mathbf{k})g^{-1}(\mathbf{k})$ for fields in the fundamental and anti-fundamental representations, respectively.

Applying these rules to (3.3), we obtain the matter part of the lattice action (2.22). Combining the result in [12], we conclude that the equivalence between the orbifolding procedure and the geometrical discretization scheme still holds for a theory with matter fields in the fundamental representation. In this scheme, the correspondence between lattice variables and continuum fields is manifest. Furthermore, we can obtain the supersymmetry transformation of the lattice variables by applying the above rule to the supersymmetry transformation (3.5), which coincides with the supersymmetry transformation obtained from (2.16) in the previous section. Since the continuum action is written in a Q-exact form as (3.4), the Q-symmetry is obviously preserved after the discretization. Thus we can conclude that the continuum limit of the lattice theory given by (2.20) is two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory coupled with hypermultiplets.

4. Conclusion and discussion

In this paper, we have constructed a lattice formulation of two-dimensional $\mathcal{N}=(2,2)$ supersymmetric $\mathrm{U}(N_c)$ gauge theory coupled with hypermultiplets in the fundamental representation. We have constructed the theory using the orbifolding procedure from the mother theory with eight supercharges by combining two kinds of orbifold projections. The obtained lattice theory preserves only one supercharge. We have also shown that the same lattice action can be constructed by extending the geometrical discretization scheme.

[6]The direction of the p-cell is determined by the $\mathrm{U}(1)$ charge of the p-form. For detail, see [46].

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This suggests that the equivalence between these two schemes to construct a supersymmetric lattice theory holds even for a theory with matter fields.

In the construction of the model (2.20), we started with a mother theory with eight supercharges while the the obtained theory is a lattice theory for two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory, which has four supercharges. This is the first example in the orbifolding procedure where the number of supercharges of the mother theory and that of the obtained (continuum) theory is different. In fact, all the lattice formulations constructed so far are those for theories with the same number of supercharges with the mother theory [1−3, 49, 8]. Although it is anticipated that the additional orbifold projection corresponding to the Z_2 transformation (2.6) would be the origin of this phenomenon, the reason is still unclear. Moreover, at present, there is no principle to use this Z_2 transformation to obtain an $\mathcal{N} = (2,2)$ theory. Even if we start with the same mother theory, we can construct several different lattice formulations with fields in the fundamental representation by changing the definition of the Z_2 transformation. It would be an interesting future work to clarify a general principle to construct a lattice formulation with matter fields that has a proper continuum limit.

Another interesting observation is that the construction we made in section 2 is quite similar to a system of intersecting D-branes. In section 2, we first introduced two lattice space-times and the matter fields come from the lattice variables living on links that connect them. Roughly speaking, we can regards the two lattice space-times as two bunches of D-branes and the link variables between them can be regarded as open strings between them. An important distinction is that we froze the degrees of freedom on the N_f-lattice by hand while it is automatic in intersecting D-branes. For example, in the case of a system of N_c D1-branes and N_f D5-branes, the gauge coupling constant on the D5-branes becomes effectively zero from the low energy effective theory point of view on the D1-branes since D5-brane is infinitely heavier than D1-brane. The same thing might occur in the orbifold construction if, for example, we could change the dimensionality of the two lattice space-times. This would also be an important future work.

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A. Lattice formulation before the restriction (2.19)

In this appendix, we write down the lattice action explicitly that is obtained by substituting (2.13) and (2.14) followed by the shift, $z_m \to 1/a + z_m$ and $\bar{z}_m \to 1/a + \bar{z}_m$. This is a lattice formulation for a two-dimensional $\mathrm{U}(N_c) \times \mathrm{U}(N_f)$ quiver gauge theory coupled with matter fields in the bi-fundamental representation. When we considered matter fields in the fundamental representation, we regarded only the N_c-lattice in figure 1 as a "real"

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lattice space-time. In the case of the quiver gauge theory, however, we have to consider both of the N_c-lattice and the N_f-lattice. Then we express a position in the N_c-lattice by an integer valued two-vector $\mathbf{k}$ while the same position in the N_f-lattice is expressed by taking an underline as $\underline{\mathbf{k}}$. Correspondingly, it is convenient to change the notation of the matter fields since we have respected only the N_c-lattice in (2.14). Instead of (2.14), we use a new notation:

$$
\begin{aligned}
\phi &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(-\sqrt{2}i\bar{\tilde{\phi}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}}) \otimes E_{\mathbf{k},N+\mathbf{k}+\mathbf{q}} + \sqrt{2}i\bar{\phi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{q}} \right), \\
\bar{\phi} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(-\sqrt{2}i\phi(\mathbf{k}+\mathbf{q}, \underline{\mathbf{k}}) \otimes E_{\mathbf{k}+\mathbf{q},N+\mathbf{k}} + \sqrt{2}i\tilde{\phi}(\underline{\mathbf{k}+\mathbf{q}}, \mathbf{k}) \otimes E_{N+\mathbf{k}+\mathbf{q},\mathbf{k}} \right), \\
\bar{\eta} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{\tilde{\psi}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}}) \otimes E_{\mathbf{k},N+\mathbf{k}+\mathbf{q}} + \bar{\psi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\mathbf{q}} \right), \\
\bar{\psi}_m &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\psi_m(\mathbf{k}+\hat{m}+\mathbf{q}, \underline{\mathbf{k}}) \otimes E_{\mathbf{k}+\hat{m}+\mathbf{q},N+\mathbf{k}} + \tilde{\psi}_m(\underline{\mathbf{k}+\hat{m}+\mathbf{q}}, \mathbf{k}) \otimes E_{N+\mathbf{k}+\hat{m}+\mathbf{q},\mathbf{k}} \right), \\
\bar{\xi}_{mn} &\equiv \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \left(\bar{\tilde{\psi}}_{mn}(\mathbf{k}, \underline{\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}}) \otimes E_{\mathbf{k},N+\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}} \right. \\
&\qquad\qquad \left. + \bar{\psi}_{mn}(\underline{\mathbf{k}}, \mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}) \otimes E_{N+\mathbf{k},\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}} \right).
\end{aligned} \tag{A.1}
$$

The fields $\phi(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}})$, $\bar{\tilde{\phi}}(\mathbf{k}+\mathbf{q}, \underline{\mathbf{k}})$, $\psi_m(\mathbf{k}+\hat{m}+\mathbf{q}, \underline{\mathbf{k}})$, $\bar{\tilde{\psi}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}})$ and $\bar{\tilde{\psi}}_{mn}(\mathbf{k}, \underline{\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}})$ are in the representation $(\square, \overline{\square})$ of $\mathrm{U}(N_c)\times\mathrm{U}(N_f)$. Correspondingly, the field $\phi(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}})$ lives on the link $(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}})$ and similar for the other fields. Similarly, $\bar{\phi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q})$, $\tilde{\phi}(\underline{\mathbf{k}+\mathbf{q}}, \mathbf{k})$, $\tilde{\psi}_m(\underline{\mathbf{k}+\hat{m}+\mathbf{q}}, \mathbf{k})$, $\bar{\psi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q})$ and $\bar{\psi}_{mn}(\underline{\mathbf{k}}, \mathbf{k}+\hat{m}+\hat{n}+\mathbf{q})$ are in the representation $(\overline{\square}, \square)$. The filed $\bar{\phi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q})$ lives on the link $(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q})$ and so on.

For a general field $\Phi(\mathbf{k}, \underline{\mathbf{l}})$ in the representation $(\square, \overline{\square})$, we define covariant differences as

$$
\begin{aligned}
\mathcal{D}_m^+\Phi(\mathbf{k}, \underline{\mathbf{l}}) &\equiv \nabla_m^+\Phi(\mathbf{k}, \underline{\mathbf{l}}) + z_m(\mathbf{k})\Phi(\mathbf{k}+\hat{m}, \underline{\mathbf{l}+\hat{m}}) - \Phi(\mathbf{k}, \underline{\mathbf{l}})\hat{z}_m(\underline{\mathbf{l}}), \\
\mathcal{D}_m^-\Phi(\mathbf{k}, \underline{\mathbf{l}}) &\equiv \nabla_m^-\Phi(\mathbf{k}, \underline{\mathbf{l}}) + z_m(\mathbf{k}-\hat{m})\Phi(\mathbf{k}, \underline{\mathbf{l}}) - \Phi(\mathbf{k}-\hat{m}, \underline{\mathbf{l}-\hat{m}})\hat{z}_m(\underline{\mathbf{l}-\hat{m}}), \\
\bar{\mathcal{D}}_m^+\Phi(\mathbf{k}, \underline{\mathbf{l}}) &\equiv \nabla_m^+\Phi(\mathbf{k}, \underline{\mathbf{l}}) + \Phi(\mathbf{k}+\hat{m}, \underline{\mathbf{l}+\hat{m}})\bar{\hat{z}}_m(\mathbf{l}) - \bar{z}_m(\mathbf{k})\Phi(\mathbf{k}, \underline{\mathbf{l}}), \\
\bar{\mathcal{D}}_m^-\Phi(\mathbf{k}, \mathbf{l}) &\equiv +\Phi(\mathbf{k}, \underline{\mathbf{l}})\bar{\hat{z}}_m(\mathbf{l}-\hat{m}) - \bar{z}_m(\mathbf{k}-\hat{m})\Phi(\mathbf{k}-\hat{m}, \underline{\mathbf{l}-\hat{m}}).
\end{aligned} \tag{A.2}
$$

Similarly, for a general field $\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l})$ in the representation $(\overline{\square}, \square)$, we define

$$
\begin{aligned}
\mathcal{D}_m^+\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) &\equiv \nabla_m^+\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) + \hat{z}_m(\underline{\mathbf{k}})\bar{\Phi}(\underline{\mathbf{k}+\hat{m}}, \mathbf{l}+\hat{m}) - \bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l})z_m(\mathbf{l}), \\
\mathcal{D}_m^-\bar{\Phi}(\mathbf{k}, \mathbf{l}) &\equiv \nabla_m^-\bar{\Phi}(\mathbf{k}, \mathbf{l}) + \hat{z}_m(\underline{\mathbf{k}-\hat{m}})\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) - \bar{\Phi}(\underline{\mathbf{k}-\hat{m}}, \mathbf{l}-\hat{m})z_m(\mathbf{l}-\hat{m}), \\
\bar{\mathcal{D}}_m^+\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) &\equiv \nabla_m^+\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) + \bar{\Phi}(\underline{\mathbf{k}+\hat{m}}, \mathbf{l}+\hat{m})\hat{z}_m(\mathbf{k}) - \bar{\hat{z}}_m(\underline{\mathbf{k}})\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}), \\
\bar{\mathcal{D}}_m^-\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) &\equiv \nabla_m^-\bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l}) + \bar{\Phi}(\underline{\mathbf{k}}, \mathbf{l})\hat{z}_m(\mathbf{k}-\hat{m}) - \bar{\hat{z}}_m(\underline{\mathbf{k}-\hat{m}})\bar{\Phi}(\underline{\mathbf{k}-\hat{m}}, \mathbf{l}-\hat{m}).
\end{aligned} \tag{A.3}
$$

Using these notations, we can write down the action of the lattice theory as

$$
S = S^{\mathrm{boson}} + S^{\mathrm{fermion}}, \tag{A.4}
$$

with

$$
\begin{aligned}
S^{\text{boson}} =& \mathrm{Tr}_{N_c} \sum_{\mathbf{k}} \Bigg(\frac{1}{4} \left|\mathcal{F}_{mn}(\mathbf{k})\right|^2 + \mathcal{D}^+_m \bar{\bar{\tilde{\phi}}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}}) \bar{\mathcal{D}}^+_m \tilde{\phi}(\underline{\mathbf{k}+\mathbf{q}}, \mathbf{k}) - \frac{1}{2} d^2(\mathbf{k}) \\
&\qquad + \left[\frac{1}{2}\mathcal{G}(\mathbf{k}) + \bar{\bar{\tilde{\phi}}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}}) \tilde{\phi}(\underline{\mathbf{k}+\mathbf{q}}, \mathbf{k}) - \phi(\mathbf{k}, \underline{\mathbf{k}-\mathbf{q}}) \bar{\phi}(\underline{\mathbf{k}-\mathbf{q}}, \mathbf{k}) \right] d(\mathbf{k}) \Bigg) \\
&\qquad + \mathrm{Tr}_{N_f} \sum_{\mathbf{k}} \Bigg(\frac{1}{4} \left|\hat{\mathcal{F}}_{\bar{m}\bar{n}}(\underline{\mathbf{k}})\right|^2 + \mathcal{D}^+_m \bar{\phi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q}) \bar{\mathcal{D}}^+_m \tilde{\phi}(\mathbf{k}+\mathbf{q}, \underline{\mathbf{k}}) - \frac{1}{2} \hat{d}^2(\mathbf{k}) \\
&\qquad + \left[\frac{1}{2}\hat{\mathcal{G}}(\underline{\mathbf{k}}) + \bar{\phi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q}) \phi(\mathbf{k}+\mathbf{q}, \underline{\mathbf{k}}) - \tilde{\phi}(\underline{\mathbf{k}}, \mathbf{k}-\mathbf{q}) \bar{\bar{\tilde{\phi}}}(\mathbf{k}-\mathbf{q}, \underline{\mathbf{k}}) \right] \hat{d}(\mathbf{k}) \Bigg), \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
S^{\text{fermion}} =& \mathrm{Tr}_{N_c} \sum_{\mathbf{k}} \Bigg(-\lambda(\mathbf{k}) \bar{\mathcal{D}}^-_m \lambda_m(\mathbf{k}) + \frac{1}{2}\lambda_{mn}(\mathbf{k}) \left(\mathcal{D}^+_m \lambda_n(\mathbf{k}) - \mathcal{D}^+_n \lambda_m(\mathbf{k}) \right) \\
& + \bar{\bar{\tilde{\psi}}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}}) \mathcal{D}^-_m \tilde{\psi}_m(\underline{\mathbf{k}+\hat{m}+\mathbf{q}}, \mathbf{k}) \\
& - \frac{1}{2} \bar{\bar{\tilde{\psi}}}_{mn}(\mathbf{k}, \underline{\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}}) \left(\bar{\mathcal{D}}^+_m \tilde{\psi}_n(\underline{\mathbf{k}+\hat{n}+\mathbf{q}}, \mathbf{k}) - \bar{\mathcal{D}}^+_n \tilde{\psi}_m(\underline{\mathbf{k}+\hat{m}+\mathbf{q}}, \mathbf{k}) \right) \\
& - \sqrt{2} i \lambda(\mathbf{k}) \big(\phi(\mathbf{k}, \underline{\mathbf{k}-\mathbf{q}}) \bar{\psi}(\underline{\mathbf{k}-\mathbf{q}}, \mathbf{k}) - \bar{\bar{\tilde{\psi}}}(\mathbf{k}, \underline{\mathbf{k}+\mathbf{q}}) \tilde{\phi}(\underline{\mathbf{k}+\mathbf{q}}, \mathbf{k}) \big) \\
& + \sqrt{2} i \lambda_m(\mathbf{k}) \big(\bar{\bar{\tilde{\phi}}}(\mathbf{k}+\hat{m}, \underline{\mathbf{k}+\hat{m}+\mathbf{q}}) \tilde{\psi}_m(\underline{\mathbf{k}+\hat{m}+\mathbf{q}}, \mathbf{k}) \\
& - \psi_m(\mathbf{k}+\hat{m}, \underline{\mathbf{k}-\mathbf{q}}) \bar{\phi}(\underline{\mathbf{k}-\mathbf{q}}, \mathbf{k}) \big) \\
& - \frac{\sqrt{2} i}{2} \lambda_{mn}(\mathbf{k}) \big(\phi(\mathbf{k}, \underline{\mathbf{k}-\mathbf{q}}) \bar{\psi}_{mn}(\underline{\mathbf{k}-\mathbf{q}}, \mathbf{k}+\hat{m}+\hat{n}) \\
& - \bar{\bar{\tilde{\psi}}}_{mn}(\mathbf{k}, \underline{\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}}) \tilde{\phi}(\underline{\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}}, \mathbf{k}+\hat{m}+\hat{n}) \big) \Bigg) \\
& + \mathrm{Tr}_{N_f} \sum_{\mathbf{k}} \Bigg(-\hat{\lambda}(\underline{\mathbf{k}}) \bar{\mathcal{D}}^-_m \hat{\lambda}_m(\underline{\mathbf{k}}) + \frac{1}{2}\hat{\lambda}_{mn}(\underline{\mathbf{k}}) \left(\mathcal{D}^+_m \hat{\lambda}_n(\underline{\mathbf{k}}) - \mathcal{D}^+_n \hat{\lambda}_m(\underline{\mathbf{k}}) \right) \\
& + \bar{\psi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q}) \mathcal{D}^-_m \psi_m(\mathbf{k}+\hat{m}+\mathbf{q}, \underline{\mathbf{k}}) \\
& - \frac{1}{2} \bar{\psi}_{mn}(\underline{\mathbf{k}}, \mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}) \left(\bar{\mathcal{D}}^+_m \psi_n(\mathbf{k}+\hat{n}+\mathbf{q}, \underline{\mathbf{k}}) - \bar{\mathcal{D}}^+_n \psi_m(\mathbf{k}+\hat{m}+\mathbf{q}, \underline{\mathbf{k}}) \right) \\
& + \sqrt{2} i \hat{\lambda}(\underline{\mathbf{k}}) \big(\tilde{\phi}(\underline{\mathbf{k}}, \mathbf{k}-\mathbf{q}) \bar{\bar{\tilde{\psi}}}(\mathbf{k}-\mathbf{q}, \underline{\mathbf{k}}) - \bar{\psi}(\underline{\mathbf{k}}, \mathbf{k}+\mathbf{q}) \phi(\mathbf{k}+\mathbf{q}, \underline{\mathbf{k}}) \big) \\
& - \sqrt{2} i \hat{\lambda}_m(\underline{\mathbf{k}}) \big(\bar{\phi}(\underline{\mathbf{k}+\hat{m}}, \mathbf{k}+\hat{m}+\mathbf{q}) \psi_m(\mathbf{k}+\hat{m}+\mathbf{q}, \underline{\mathbf{k}}) \\
& - \tilde{\psi}_m(\underline{\mathbf{k}+\hat{m}}, \mathbf{k}-\mathbf{q}) \bar{\bar{\tilde{\phi}}}(\mathbf{k}-/\mathbf{q}, \underline{\mathbf{k}}) \big) \\
& + \frac{\sqrt{2} i}{2} \hat{\lambda}_{mn}(\underline{\mathbf{k}}) \big(\tilde{\phi}(\underline{\mathbf{k}}, \mathbf{k}-\mathbf{q}) \bar{\bar{\tilde{\psi}}}_{mn}(\mathbf{k}-\mathbf{q}, \underline{\mathbf{k}+\hat{m}+\hat{n}}) \\
& - \bar{\psi}_{mn}(\underline{\mathbf{k}}, \mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}) \phi(\mathbf{k}+\hat{m}+\hat{n}+\mathbf{q}, \underline{\mathbf{k}+\hat{m}+\hat{n}}) \big) \Bigg), \qquad \text{(A.6)}
\end{aligned}
$$

where $\mathcal{F}_{mn}(\mathbf{k})$, $\hat{\mathcal{F}}_{mn}(\underline{\mathbf{k}})$, $\mathcal{G}(\mathbf{k})$ and $\hat{\mathcal{G}}(\underline{(}\mathbf{k}))$ are defined by

$$
\begin{aligned}
\mathcal{F}_{mn}(\mathbf{k}) &\equiv \nabla^+_m z_n(\mathbf{k}) - \nabla^+_n z_m(\mathbf{k}) + z_m(\mathbf{k}) z_n(\mathbf{k}+\hat{m}) - z_n(\mathbf{k}) z_m(\mathbf{k}+\hat{n}), \\
\hat{\mathcal{F}}_{mn}(\underline{\mathbf{k}}) &\equiv \nabla^+_m \hat{z}_n(\underline{\mathbf{k}}) - \nabla^+_n \hat{z}_m(\underline{\mathbf{k}}) + \hat{z}_m(\underline{\mathbf{k}}) \hat{z}_n(\underline{\mathbf{k}+\hat{m}}) - \hat{z}_n(\underline{\mathbf{k}}) \hat{z}_m(\underline{\mathbf{k}+\hat{n}}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}(\mathbf{k}) &\equiv \sum_m \left(\nabla_m^- \left(z_m(\mathbf{k}) + \bar{z}_m(\mathbf{k})\right) + z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\hat{m})z_m(\mathbf{k}-\hat{m})\right), \\
\hat{\mathcal{G}}(\underline{\mathbf{k}}) &\equiv \sum_m \left(\nabla_m^- \left(\hat{z}_m(\underline{\mathbf{k}}) + \bar{\hat{z}}_m(\underline{\mathbf{k}})\right) + \hat{z}_m(\underline{\mathbf{k}})\bar{\hat{z}}_m(\underline{\mathbf{k}}) - \bar{\hat{z}}_m(\underline{\mathbf{k}-\hat{m}})\hat{z}_m(\underline{\mathbf{k}-\hat{m}})\right). \qquad \text{(A.7)}
\end{aligned}
$$

References

[1] A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a Euclidean spacetime lattice. I: a target theory with four supercharges*, *JHEP* **08** (2003) 024 [hep-lat/0302017].

[2] A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a Euclidean spacetime lattice. II: target theories with eight supercharges*, *JHEP* **12** (2003) 031 [hep-lat/0307012].

[3] D.B. Kaplan and M. Ünsal, *A Euclidean lattice construction of supersymmetric Yang-Mills theories with sixteen supercharges*, *JHEP* **09** (2005) 042 [hep-lat/0503039].

[4] S. Catterall, *Simulations of $N = 2$ super Yang-Mills theory in two dimensions*, *JHEP* **03** (2006) 032 [hep-lat/0602004].

[5] J. Giedt, *Non-positive fermion determinants in lattice supersymmetry*, *Nucl. Phys.* **B 668** (2003) 138 [hep-lat/0304006].

[6] T. Onogi and T. Takimi, *Perturbative study of the supersymmetric lattice theory from matrix model*, *Phys. Rev.* **D 72** (2005) 074504 [hep-lat/0506014].

[7] K. Ohta and T. Takimi, *Lattice formulation of two dimensional topological field theory*, *Prog. Theor. Phys.* **117** (2007) 317 [hep-lat/0611011].

[8] P.H. Damgaard and S. Matsuura, *Classification of supersymmetric lattice gauge theories by orbifolding*, *JHEP* **07** (2007) 051 [arXiv:0704.2696].

[9] S. Catterall, *Lattice supersymmetry and topological field theory*, *JHEP* **05** (2003) 038 [arXiv:0301028].

[10] S. Catterall, *A geometrical approach to $N = 2$ super Yang-Mills theory on the two dimensional lattice*, *JHEP* **11** (2004) 006 [hep-lat/0410052].

[11] S. Catterall, *Lattice formulation of $N = 4$ super Yang-Mills theory*, *JHEP* **06** (2005) 027 [hep-lat/0503036].

[12] S. Catterall, *From twisted supersymmetry to orbifold lattices*, *JHEP* **01** (2008) 048 [arXiv:0712.2532].

[13] F. Sugino, *A lattice formulation of super Yang-Mills theories with exact supersymmetry*, *JHEP* **01** (2004) 015 [hep-lat/0311021].

[14] F. Sugino, *Super Yang-Mills theories on the two-dimensional lattice with exact supersymmetry*, *JHEP* **03** (2004) 067 [hep-lat/0401017].

[15] F. Sugino, *Various super Yang-Mills theories with exact supersymmetry on the lattice*, *JHEP* **01** (2005) 016 [hep-lat/0410035].

[16] F. Sugino, *Two-dimensional compact $\mathcal{N} = (2, 2)$ lattice super Yang-Mills theory with exact supersymmetry*, *Phys. Lett.* **B 635** (2006) 218 [hep-lat/0601024].

[17] H. Suzuki, *Two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory on computer*, *JHEP* **09** (2007) 052 [arXiv:0706.1392].

JHEP07(2008)127

[18] I. Kanamori, H. Suzuki and F. Sugino, *Euclidean lattice simulation for the dynamical supersymmetry breaking*, *Phys. Rev.* **D 77** (2008) 091502 [arXiv:0711.2099].

[19] I. Kanamori, F. Sugino and H. Suzuki, *Observing dynamical supersymmetry breaking with euclidean lattice simulations*, *Prog. Theor. Phys.* **119** (2008) 797 [arXiv:0711.2132].

[20] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Twisted superspace on a lattice*, *Nucl. Phys.* **B 707** (2005) 100 [hep-lat/0406029].

[21] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Exact extended supersymmetry on a lattice: twisted $N = 2$ super Yang-Mills in two dimensions*, *Phys. Lett.* **B 633** (2006) 645 [hep-lat/0507029].

[22] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Exact extended supersymmetry on a lattice: twisted $N = 4$ super Yang-Mills in three dimensions*, *Nucl. Phys.* **B 798** (2008) 168 [arXiv:0707.3533].

[23] K. Nagata and Y.-S. Wu, *Twisted SUSY invariant formulation of Chern-Simons gauge theory on a lattice*, arXiv:0803.4339.

[24] F. Bruckmann and M. de Kok, *Noncommutativity approach to supersymmetry on the lattice: SUSY quantum mechanics and an inconsistency*, *Phys. Rev.* **D 73** (2006) 074511 [hep-lat/0603003].

[25] F. Bruckmann, S. Catterall and M. de Kok, *A critique of the link approach to exact lattice supersymmetry*, *Phys. Rev.* **D 75** (2007) 045016 [hep-lat/0611001].

[26] S. Arianos, A. D'Adda, N. Kawamoto and J. Saito, *Lattice supersymmetry in 1D with two supercharges*, PoS(LATTICE 2007)259 [arXiv:0710.0487].

[27] K. Nagata, *Exact lattice supersymmetry at large-N*, arXiv:0805.4235.

[28] M. Ünsal, *Supersymmetric deformations of type IIB matrix model as matrix regularization of $N = 4$ SYM*, *JHEP* **04** (2006) 002 [hep-th/0510004].

[29] J. Nishimura, *Four-dimensional $N = 1$ supersymmetric Yang-Mills theory on the lattice without fine-tuning*, *Phys. Lett.* **B 406** (1997) 215 [hep-lat/9701013].

[30] D.B. Kaplan, *Dynamical generation of supersymmetry*, *Phys. Lett.* **B 136** (1984) 162.

[31] N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine-tuning*, *Int. J. Mod. Phys.* **A 13** (1998) 2841 [hep-th/9705152].

[32] H. Neuberger, *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D 57** (1998) 5417 [hep-lat/9710089].

[33] D.B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543 [hep-lat/0002030].

[34] G.T. Fleming, J.B. Kogut and P.M. Vranas, *Super Yang-Mills on the lattice with domain wall fermions*, *Phys. Rev.* **D 64** (2001) 034510 [hep-lat/0008009].

[35] I. Montvay, *Supersymmetric Yang-Mills theory on the lattice*, *Int. J. Mod. Phys.* **A 17** (2002) 2377 [hep-lat/0112007].

[36] H. Suzuki and Y. Taniguchi, *Two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory on the lattice via dimensional reduction*, *JHEP* **10** (2005) 082 [hep-lat/0507019].

[37] J.W. Elliott and G.D. Moore, *Three dimensional $N = 2$ supersymmetry on the lattice*, PoS(LAT2005)245 [hep-lat/0509032].

[38] J.W. Elliott and G.D. Moore, *3D $N = 1$ SYM Chern-Simons theory on the lattice*, *JHEP* **11** (2007) 067 [arXiv:0708.3214].

[39] M. Hanada, J. Nishimura and S. Takeuchi, *Non-lattice simulation for supersymmetric gauge theories in one dimension*, *Phys. Rev. Lett.* **99** (2007) 161602 [arXiv:0706.1647].

[40] K.N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, *Monte Carlo studies of supersymmetric matrix quantum mechanics with sixteen supercharges at finite temperature*, *Phys. Rev. Lett.* **100** (2008) 021601 [arXiv:0707.4454].

[41] N. Kawahara, J. Nishimura and S. Takeuchi, *Phase structure of matrix quantum mechanics at finite temperature*, *JHEP* **10** (2007) 097 [arXiv:0706.3517].

[42] N. Kawahara, J. Nishimura and S. Takeuchi, *High temperature expansion in supersymmetric matrix quantum mechanics*, *JHEP* **12** (2007) 103 [arXiv:0710.2188].

[43] S. Catterall and T. Wiseman, *Towards lattice simulation of the gauge theory duals to black holes and hot strings*, *JHEP* **12** (2007) 104 [arXiv:0706.3518].

[44] S. Catterall and T. Wiseman, *Black hole thermodynamics from simulations of lattice Yang-Mills theory*, arXiv:0803.4273.

[45] P.H. Damgaard and S. Matsuura, *Relations among supersymmetric lattice gauge theories via orbifolding*, *JHEP* **08** (2007) 087 [arXiv:0706.3007].

[46] P.H. Damgaard and S. Matsuura, *Geometry of orbifolded supersymmetric lattice gauge theories*, *Phys. Lett.* **B 661** (2008) 52 [arXiv:0801.2936].

[47] T. Takimi, *Relationship between various supersymmetric lattice models*, *JHEP* **07** (2007) 010 [arXiv:0705.3831].

[48] P.H. Damgaard and S. Matsuura, *Lattice supersymmetry: equivalence between the link approach and orbifolding*, *JHEP* **09** (2007) 097 [arXiv:0708.4129].

[49] M.G. Endres and D.B. Kaplan, *Lattice formulation of $(2, 2)$ supersymmetric gauge theories with matter fields*, *JHEP* **10** (2006) 076 [hep-lat/0604012].

[50] J. Giedt, *Quiver lattice supersymmetric matter, $D1/D5$ branes and AdS_3/CFT_2*, hep-lat/0605004.

[51] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1983).